Name ____________________ Class __________ Date __________

3.3 Finding Complex Solutions of Quadratic Equations

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Essential Question: How can you find the complex solutions of any quadratic equation?

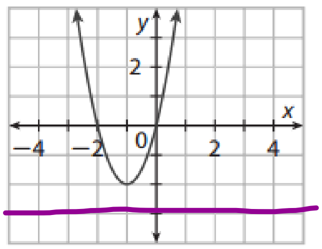
Resource Locker

Explore Investigating Real Solutions of Quadratic Equations

(A) Complete the table.

$ax^2 + bx + c = 0$	$ax^2 + bx = -c$	$f(x) = ax^2 + bx$	$g(x) = -c$
$2x^2 + 4x + 1 = 0$			
$2x^2 + 4x + 2 = 0$			
$2x^2 + 4x + 3 = 0$			

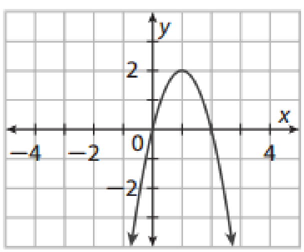
(B) The graph of $f(x) = 2x^2 + 4x$ is shown. Graph each $g(x)$. Complete the table.



Equation	Number of Real Solutions
$2x^2 + 4x + 1 = 0$	
$2x^2 + 4x + 2 = 0$	
$2x^2 + 4x + 3 = 0$	

(C) Repeat Steps A and B when $f(x) = -2x^2 + 4x$.

$ax^2 + bx + c = 0$	$ax^2 + bx = -c$	$f(x) = ax^2 + bx$	$g(x) = -c$
$-2x^2 + 4x - 1 = 0$			
$-2x^2 + 4x - 2 = 0$			
$-2x^2 + 4x - 3 = 0$			



Equation	Number of Real Solutions
$-2x^2 + 4x - 1 = 0$	
$-2x^2 + 4x - 2 = 0$	
$-2x^2 + 4x - 3 = 0$	

Reflect

1. Look back at Steps A and B. Notice that the minimum value of $f(x)$ in Steps A and B is -2. Complete the table by identifying how many real solutions the equation $f(x) = g(x)$ has for the given values of $g(x)$.

Value of $g(x)$	Number of Real Solutions of $f(x) = g(x)$
$g(x) = -2$	
$g(x) > -2$	
$g(x) < -2$	

2. Look back at Step C. Notice that the maximum value of $f(x)$ in Step C is 2. Complete the table by identifying how many real solutions the equation $f(x) = g(x)$ has for the given values of $g(x)$.

Value of $g(x)$	Number of Real Solutions of $f(x) = g(x)$
$g(x) = 2$	
$g(x) > 2$	
$g(x) < 2$	

3. You can generalize Reflect 1: For $f(x) = ax^2 + bx$ where $a > 0$, $f(x) = g(x)$ where $g(x) = -c$ has real solutions when $g(x)$ is greater than or equal to the minimum value of $f(x)$. The minimum value of $f(x)$ is

$$f\left(-\frac{b}{2a}\right) = a\left(-\frac{b}{2a}\right)^2 + b\left(-\frac{b}{2a}\right) = a\left(\frac{b^2}{4a^2}\right) - \frac{b^2}{2a} = \frac{b^2}{4a} - \frac{b^2}{2a} = \frac{b^2}{4a} - \frac{2b^2}{4a} = -\frac{b^2}{4a}.$$

So, $f(x) = g(x)$ has real solutions when $g(x) \geq -\frac{b^2}{4a}$.

Substitute $-c$ for $g(x)$. $-c \geq -\frac{b^2}{4a}$

Add $\frac{b^2}{4a}$ to both sides. $\frac{b^2}{4a} - c \geq 0$

Multiply both sides by $4a$, which is positive. $b^2 - 4ac \geq 0$

In other words, the equation $ax^2 + bx + c = 0$ where $a > 0$ has real solutions when $b^2 - 4ac \geq 0$.

Generalize the results of Reflect 2 in a similar way. What do you notice?

Explain 1 Finding Complex Solutions by Completing the Square

Recall that completing the square for the expression $x^2 + bx$ requires adding $\left(\frac{b}{2}\right)^2$ to it, resulting in the perfect square trinomial $x^2 + bx + \left(\frac{b}{2}\right)^2$, which you can factor as $\left(x + \frac{b}{2}\right)^2$. Don't forget that when $x^2 + bx$ appears on one side of an equation, adding $\left(\frac{b}{2}\right)^2$ to it requires adding $\left(\frac{b}{2}\right)^2$ to the other side as well.

Example 1 **Solve the equation by completing the square. State whether the solutions are real or non-real.**

(A) $3x^2 + 9x - 6 = 0$

1. Write the equation in the form $x^2 + bx = c$.

$3x^2 + 9x - 6 = 0$

$3x^2 + 9x = 6$

$x^2 + 3x = 2$

A B

$x^2 + 3x + \frac{9}{4} = 2 + \frac{9}{4}$

2. Identify b and $\left(\frac{b}{2}\right)^2$.

$\left(\frac{3}{2}\right)^2$

$b = 3$

$\left(\frac{b}{2}\right)^2 = \left(\frac{3}{2}\right)^2 = \frac{9}{4}$

3. Add $\left(\frac{b}{2}\right)^2$ to both sides of the equation.

$x^2 + 3x + \frac{9}{4} = 2 + \frac{9}{4}$

$(x+\frac{3}{2})(x+\frac{3}{2})$

$(x+\frac{3}{2})^2$

4. Solve for x.

$\left(x + \frac{3}{2}\right)^2 = 2 + \frac{9}{4}$

$\left(x + \frac{3}{2}\right)^2 = \frac{17}{4}$

$x + \frac{3}{2} = \pm\sqrt{\frac{17}{4}}$

$x + \frac{3}{2} = \pm\frac{\sqrt{17}}{2}$

$x = -\frac{3}{2} \pm \frac{\sqrt{17}}{2}$

$x = \frac{-3 \pm \sqrt{17}}{2}$

There are two real solutions: $\frac{-3 + \sqrt{17}}{2}$ and $\frac{-3 - \sqrt{17}}{2}$.

(B) $x^2 - 2x + 7 = 0$

1. Write the equation in the form $x^2 + bx = c$.

$x^2 - 2x + 1 = -7 + 1$

2. Identify b and $\left(\frac{b}{2}\right)^2$.

$b = \boxed{-2}$

$\left(\frac{b}{2}\right)^2 = \left(\frac{\boxed{-2}}{2}\right)^2 = \boxed{1}$

3. Add $\left(\frac{b}{2}\right)^2$ to both sides.

$x^2 - 2x + \boxed{1} = -7 + \boxed{1}$

$(x-1)(x-1)$

4. Solve for x.

$x^2 - 2x + \boxed{1} = -7 + \boxed{1}$

$\left(x - \boxed{1}\right)^2 = -6$

$x - \boxed{1} = \pm\sqrt{-6}$

$x = 1 \pm \sqrt{-6}$

$x = 1 \pm i\sqrt{6}$

There are two real/non-real solutions: ________

and ________.

Reflect

4. How many complex solutions do the equations in Parts A and B have? Explain.

Your Turn

Solve the equation by completing the square. State whether the solutions are real or non-real.

5. $x^2 + 8x + 17 = 0$

$x^2+8x+16=-17+16$

$(x+4)(x+4)=-1$

$(x+4)^2=-1$

$x+4=\pm\sqrt{-1}$

$x=-4\pm i$

6. $x^2 + 10x - 7 = 0$

$x^2+10x+25=7+25$

$(x+5)(x+5)$

$(x+5)^2=32$

$x+5=\pm\sqrt{32}$

$x=\pm\sqrt{32}-5$

$x=-5\pm4\sqrt{2}$

$x=-5\pm4\sqrt{2}$

Explain 2 Identifying Whether Solutions Are Real or Non-real

By completing the square for the general quadratic equation $ax^2 + bx + c = 0$, you can obtain the *quadratic formula*, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, which gives the solutions of the general quadratic equation. In the quadratic formula, the expression under the radical sign, $b^2 - 4ac$, is called the *discriminant*, and its value determines whether the solutions of the quadratic equation are real or non-real.

Value of Discriminant	Number and Type of Solutions
$b^2 - 4ac > 0$	Two real solutions
$b^2 - 4ac = 0$	One real solution
$b^2 - 4ac < 0$	Two non-real solutions

Example 2 **Answer the question by writing an equation and determining whether the solutions of the equation are real or non-real.**

(A) A ball is thrown in the air with an initial vertical velocity of 14 m/s from an initial height of 2 m. The ball's height h (in meters) at time t (in seconds) can be modeled by the quadratic function $h(t) = -4.9t^2 + 14t + 2$. Does the ball reach a height of 12 m?

Set $h(t)$ equal to 12. $-4.9t^2 + 14t + 2 = 12$

A B −C

Subtract 12 from both sides. $-4.9t^2 + 14t - 10 = 0$

Find the value of the discriminant. $14^2 - 4(-4.9)(-10) = 196 - 196 = 0$

b² A C

Because the discriminant is zero, the equation has one real solution, so the ball does reach a height of 12 m.

(B) A person wants to create a vegetable garden and keep the rabbits out by enclosing it with 100 feet of fencing. The area of the garden is given by the function $A(w) = w(50 - w)$ where w is the width (in feet) of the garden. Can the garden have an area of 700 ft^2?

Set $A(w)$ equal to 700. $w(50 - w) = \boxed{700}$

Multiply on the left side. $50w - w^2 = \boxed{700}$

Subtract 700 from both sides. $\overset{A}{-w^2} + \overset{B}{50w} - \overset{C}{\boxed{700}} = 0$

$b^2 - 4AC$

Find the value of the discriminant.

$50^2 - 4(-1)(-700)$

Because the discriminant is [positive/zero/negative], the equation has [two real/one real/two non-real] solutions, so the garden [can/cannot] have an area of 700 ft^2.

Your Turn

Answer the question by writing an equation and determining if the solutions are real or non-real.

7. A hobbyist is making a toy sailboat. For the triangular sail, she wants the height h (in inches) to be twice the length of the base b (in inches). Can the area of the sail be 10 in^2?

Explain 3 Finding Complex Solutions Using the Quadratic Formula

When using the quadratic formula to solve a quadratic equation, be sure the equation is in the form $ax^2 + bx + c = 0$.

Example 3 **Solve the equation using the quadratic formula. Check a solution by substitution.**

(A) $\overset{A}{-5}x^2 \overset{B}{-2}x \overset{C}{-8} = 0$

Write the quadratic formula. $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Substitute values. $= \dfrac{-(-2) \pm \sqrt{(-2)^2 - 4(-5)(-8)}}{2(-5)}$

Simplify. $= \dfrac{2 \pm \sqrt{-156}}{-10} = \dfrac{1 \pm i\sqrt{39}}{-5}$

$\sqrt{156}$

$\sqrt{4}\ \sqrt{39}$

$2\sqrt{39}$

$\dfrac{2 \pm 2i\sqrt{39}}{-10}$

So, the two solutions are $-\frac{1}{5} - \frac{i\sqrt{39}}{5}$ and $-\frac{1}{5} + \frac{i\sqrt{39}}{5}$.

Check by substituting one of the values.

Substitute. $-5\left(-\frac{1}{5} - \frac{i\sqrt{39}}{5}\right)^2 - 2\left(-\frac{1}{5} - \frac{i\sqrt{39}}{5}\right) - 8 \stackrel{?}{=} 0$

Square. $-5\left(\frac{1}{25} + \frac{2i\sqrt{39}}{25} - \frac{39}{25}\right) - 2\left(-\frac{1}{5} - \frac{i\sqrt{39}}{5}\right) - 8 \stackrel{?}{=} 0$

Distribute. $-\frac{1}{5} - \frac{2i\sqrt{39}}{5} + \frac{39}{5} + \frac{2}{5} + \frac{2i\sqrt{39}}{5} - 8 \stackrel{?}{=} 0$

Simplify. $\frac{40}{5} - 8 \stackrel{?}{=} 0$

$0 = 0$

(B) $7x^2 + 2x + 3 = -1$

Write the equation with 0 on one side. $\overset{A}{7x^2} + \overset{B}{2x} + \overset{C}{\boxed{4}} = 0$

Write the quadratic formula. $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Substitute values. $= \frac{-\boxed{2} \pm \sqrt{\left(\boxed{2}\right)^2 - 4\left(\boxed{7}\right)\left(\boxed{4}\right)}}{2\left(\boxed{7}\right)}$

Simplify. $= \frac{-\boxed{2} \pm \sqrt{-108}}{14}$ $\rightarrow$ $\sqrt{108}$, $\sqrt{36}\sqrt{3}$, $6\sqrt{3}$

$= \frac{-\boxed{2} \pm \boxed{6}\, i\sqrt{\boxed{3}}}{14} = \frac{-\boxed{1} \pm \boxed{3}\, i\sqrt{\boxed{3}}}{7}$

So, the two solutions are ________________ and ________________.

Check by substituting one of the values.

Substitute.

Square.

Distribute.

Simplify.

Your Turn

Solve the equation using the quadratic formula. Check a solution by substitution.

8. $6x^2 - 5x - 4 = 0$

9. $x^2 + 8x + 12 = 2x$

$$= \frac{5 \pm \sqrt{(5)^2 - 4(6)(-4)}}{2(6)}$$

$$\frac{5 + \sqrt{121}}{12}$$

$$\frac{5+11}{12} \qquad \frac{5-11}{12}$$

$$16/12 \qquad -6/12$$

$$\boxed{4/3 \text{ and } -1/2}$$

Elaborate

10. Discussion Suppose that the quadratic equation $ax^2 + bx + c = 0$ has $p + qi$ where $q \neq 0$ as one of its solutions. What must the other solution be? How do you know?

11. Discussion You know that the graph of the quadratic function $f(x) = ax^2 + bx + c$ has the vertical line $x = -\frac{b}{2a}$ as its axis of symmetry. If the graph of $f(x)$ crosses the x-axis, where do the x-intercepts occur relative to the axis of symmetry? Explain.

12. Essential Question Check-In Why is using the quadratic formula to solve a quadratic equation easier than completing the square?

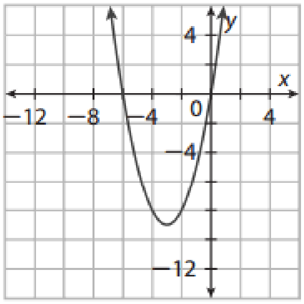
Pg 146 3-6, 9-12, 14, 17-19, 21

$b^2 - 4AC$

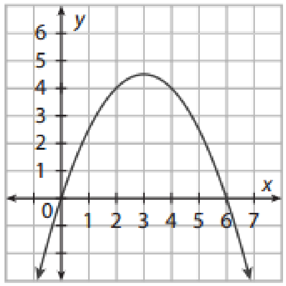
Evaluate: Homework and Practice

Personal Math Trainer
- Online Homework
- Hints and Help
- Extra Practice

1. The graph of $f(x) = x^2 + 6x$ is shown. Use the graph to determine how many real solutions the following equations have: $x^2 + 6x + 6 = 0$, $x^2 + 6x + 9 = 0$, and $x^2 + 6x + 12 = 0$. Explain.



2. The graph of $f(x) = -\frac{1}{2}x^2 + 3x$ is shown. Use the graph to determine how many real solutions the following equations have: $-\frac{1}{2}x^2 + 3x - 3 = 0$, $-\frac{1}{2}x^2 + 3x - \frac{9}{2} = 0$, and $-\frac{1}{2}x^2 + 3x - 6 = 0$. Explain.



Solve the equation by completing the square. State whether the solutions are real or non-real.

3. $x^2 + 4x + 1 = 0$

$x^2+4x+4=-1+4$

$(x+2)^2 = 3$

$x+2 = \pm\sqrt{3}$

$x = -2\pm\sqrt{3}$

4. $x^2 + 2x + 8 = 0$

$x^2+2x+1 = -8+1$

$(x+1)^2 = -7$

$x+1 = \pm\sqrt{-7}$

$x = -1\pm i\sqrt{7}$

5. $x^2 - 5x = -20$

$x^2 - 5x + \frac{25}{4} = -20 + \frac{25}{4}$

$\left(x - \frac{5}{2}\right)^2 = \frac{-55}{4}$

$x - \frac{5}{2} = \pm\sqrt{\frac{-55}{4}}$

$x = \frac{5}{2} \pm \frac{i\sqrt{55}}{2}$

6. $5x^2 - 6x = 8$

$x^2 - \frac{6}{5}x + \frac{9}{25} = \frac{8}{5} + \frac{9}{25}$

$\left(x - \frac{3}{5}\right)^2 = \frac{49}{25}$

$x - \frac{3}{5} = \pm\sqrt{\frac{49}{25}}$

$x = \frac{3}{5} \pm \frac{7}{5}$

$x = 2, -\frac{4}{5}$

7. $7x^2 + 13x = 5$

8. $-x^2 - 6x - 11 = 0$

$b^2 - 4AC$

Without solving the equation, state the number of solutions and whether they are real or non-real.

9. $-16x^2 + 4x + 13 = 0$

$4^2 - 4(-16)(13)$

848 2 Real

10. $7x^2 - 11x + 10 = 0$

$(-11)^2 - 4(7)(10)$

-159 2 Non Real

11. $-x^2 - \frac{2}{5}x = 1$

$-x^2 - \frac{2}{5}x - 1 = 0$

$\left(-\frac{2}{5}\right)^2 - 4(-1)(-1)$

$-\frac{96}{25}$ 2 Non Real

12. $4x^2 + 9 = 12x$

$4x^2 - 12x + 9 = 0$

$(-12)^2 - 4(4)(9)$

0 1 Real

Answer the question by writing an equation and determining whether the solutions of the equation are real or non-real.

13. A gardener has 140 feet of fencing to put around a rectangular vegetable garden. The function $A(w) = 70w - w^2$ gives the garden's area A (in square feet) for any width w (in feet). Does the gardener have enough fencing for the area of the garden to be 1300 ft²?

$b^2 - 4AC$

14. A golf ball is hit with an initial vertical velocity of ~~64~~ ft/s. The function $h(t) = -16t^2 + 64t$ models the height h (in feet) of the golf ball at time t (in seconds). Does the golf ball reach a height of 60 ft?

$60 \stackrel{?}{=} -16t^2 + 64t$

$0 = \overset{A}{-16}t^2 + \overset{B}{64}t \overset{C}{-60}$

$64^2 - 4(-16)(-60) = 256$

Yes

15. As a decoration for a school dance, the student council creates a parabolic arch with balloons attached to it for students to walk through as they enter the dance. The shape of the arch is modeled by the equation $y = x(5 - x)$, where x and y are measured in feet and where the origin is at one end of the arch. Can a student who is 6 feet 6 inches tall walk through the arch without ducking?

16. A small theater company currently has 200 subscribers who each pay $120 for a season ticket. The revenue from season-ticket subscriptions is $24,000. Market research indicates that for each $10 increase in the cost of a season ticket, the theater company will lose 10 subscribers. A model for the projected revenue R (in dollars) from season-ticket subscriptions is $R(p) = (120 + 10p)(200 - 10p)$, where p is the number of $10 price increases. According to this model, is it possible for the theater company to generate $25,600 in revenue by increasing the price of a season ticket?

Solve the equation using the quadratic formula. ~~Check a solution by substitution.~~

17. $x^2 - 8x + 27 = 0$

A = 1 B = -8 C = 27

$$\frac{8 \pm \sqrt{(-8)^2 - 4(1)(27)}}{2(1)}$$

$\sqrt{4}\sqrt{11}$

$$\frac{8 \pm \sqrt{-44}}{2}$$

$$\frac{8 \pm 2i\sqrt{11}}{2}$$

$4 \pm i\sqrt{11}$

18. $x^2 - 30x + 50 = 0$

A = 1 B = -30 C = 50

$$\frac{30 \pm \sqrt{(-30)^2 - 4(1)(50)}}{2(1)}$$

$\sqrt{100}\sqrt{7}$

$$\frac{30 \pm \sqrt{700}}{2}$$

$$\frac{30 \pm 10\sqrt{7}}{2}$$

$15 \pm 5\sqrt{7}$

19. $x + 3 = x^2$

$0 = x^2 - x - 3$

$A = 1 \quad B = -1 \quad C = -3$

$$\frac{1 \pm \sqrt{(-1)^2 - 4(1)(-3)}}{2(1)}$$

$$\frac{1 \pm \sqrt{13}}{2}$$

20. $2x^2 + 7 = 4x$

$b^2 - 4AC$

21. Place an X in the appropriate column of the table to classify each equation by the number and type of its solutions.

Equation	Two Real Solutions +	One Real Solution 0	Two Non-Real Solutions −
$x^2 - 3x + 1 = 0$	✓		
$x^2 - 2x + 1 = 0$		✓	
$x^2 - x + 1 = 0$			✓
$x^2 + 1 = 0$			✓
$x^2 + x + 1 = 0$			✓
$x^2 + 2x + 1 = 0$		✓	
$x^2 + 3x + 1 = 0$	✓		