Solution Section 2.8 – Related Rates

Exercise

If $y = x^2$ and $\frac{dx}{dt} = 3$, then what is $\frac{dy}{dt}$ *when* $x = -1$

Solution

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$
$$= 2x(3)$$
$$= 6x$$

$$\left.\frac{dy}{dt}\right|_{x=-1} = 6(-1) \underline{= -6}$$

Exercise

If $x = y^3 - y$ and $\frac{dy}{dt} = 5$, then what is $\frac{dx}{dt}$ *when* $y = 2$

Solution

$$\frac{dx}{dt} = \frac{dx}{dy}\frac{dy}{dt}$$
$$= \left(3y^2 - 1\right)(5)$$
$$= 5\left(3y^2 - 1\right)$$

$$\left.\frac{dx}{dt}\right|_{y=2} = 5\left(3(2)^2 - 1\right) = \underline{55}$$

Exercise

A cube's surface area increases at the rate of 72 $in^2/\sec$. At what rate is the cube's volume changing when the edge length is $x = 3$ *in*?

Solution

Cube's surface: $S = 6x^2$

$$\frac{dS}{dt} = 12x\frac{dx}{dt}$$

$$72 = 12x(3) \quad \Rightarrow \quad \underline{x = \frac{72}{26} = 2}$$

$$V = x^3 \quad \Rightarrow \quad \frac{dV}{dt} = 3x^2\frac{dx}{dt}$$

$$\left.\frac{dV}{dt}\right|_{x=3} = 3(3)^2(2) = \underline{54\ in^2/\sec}$$

Exercise

The radius r and height h of a right circular cone are related to the cone's volume V by the equation $V = \frac{1}{3}\pi r^2 h$.

a) How is $\frac{dV}{dt}$ related to $\frac{dh}{dt}$ if $\boldsymbol{r}$ is constant?

b) How is $\frac{dV}{dt}$ related to $\frac{dr}{dt}$ if $\boldsymbol{h}$ is constant?

c) How is $\frac{dV}{dt}$ related to $\frac{dr}{dt}$ and $\frac{dh}{dt}$ if neither $\boldsymbol{r}$ nor $\boldsymbol{h}$ is constant?

Solution

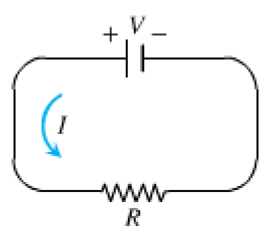
a) $\frac{dV}{dt} = \frac{1}{3}\pi r^2 \frac{dh}{dt}$

b) $\frac{dV}{dt} = \frac{2}{3}\pi rh \frac{dr}{dt}$

c) $\frac{dV}{dt} = \frac{2}{3}\pi rh \frac{dr}{dt} + \frac{1}{3}\pi r^2 \frac{dh}{dt}$

Exercise

The voltage $\boldsymbol{V}$ (volts), current $\boldsymbol{I}$ (amperes), and resistance $\boldsymbol{R}$ (ohms) of an electric circuit like the one shown here are related by the equation $V = IR$. Suppose that V is increasing at the rate of 1 volt/sec while I is decreasing at the rate of $\frac{1}{3}$ *amp* / sec. Let t denote time in seconds.



a) What is the value of $\frac{dV}{dt}$?

b) What is the value of $\frac{dI}{dt}$?

c) What equation relates $\frac{dR}{dt}$ to $\frac{dV}{dt}$ and $\frac{dI}{dt}$?

d) Find the rate at which R is changing when $V = 12$ volts and $I = 2$ amp. Is R increasing or decreasing?

Solution

a) $\frac{dV}{dt} = 1 \ volt/\sec$

b) $\frac{dI}{dt} = \frac{1}{3} \ amp/\sec$

c) $\frac{dV}{dt} = R\frac{dI}{dt} + I\frac{dR}{dt}$

$I\frac{dR}{dt} = \frac{dV}{dt} - R\frac{dI}{dt}$ $\qquad V = IR \Rightarrow R = \frac{V}{I}$

$\frac{dR}{dt} = \frac{1}{I}\left(\frac{dV}{dt} - \frac{V}{I}\frac{dI}{dt}\right)$

d) $\frac{dR}{dt} = \frac{1}{2}\left((1) - \frac{12}{2}\left(-\frac{1}{3}\right)\right) = \frac{1}{2}(3) = \frac{3}{2} \ ohms/\sec$ $\quad$ ***R*** is increasing

Exercise

Let x and y be differentiable functions of t and let $s=\sqrt{x^2+y^2}$ be the distance between the points $(x, 0)$ and $(0, y)$ in the xy–plane.

a) How is $\frac{ds}{dt}$ related to $\frac{dx}{dt}$ if y is constant?

b) How is $\frac{ds}{dt}$ related to $\frac{dx}{dt}$ and $\frac{dy}{dt}$ if neither x nor y is constant?

c) How is $\frac{dx}{dt}$ related to $\frac{dy}{dt}$ if s is constant?

Solution

$$s=\sqrt{x^2+y^2}=\left(x^2+y^2\right)^{1/2}$$

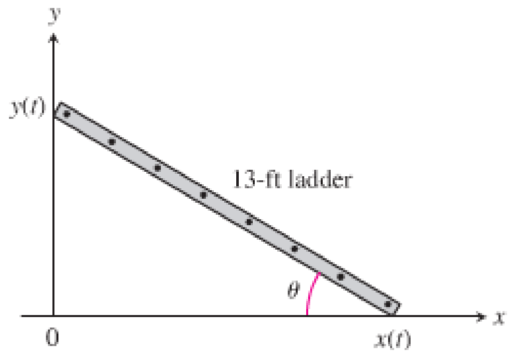
a) $$\frac{ds}{dt}=\frac{1}{2}\left(x^2+y^2\right)^{-1/2}\left(2x\frac{dx}{dt}\right)$$
$$=\frac{x}{\sqrt{x^2+y^2}}\frac{dx}{dt}$$

b) $$\frac{ds}{dt}=\frac{1}{2}\left(x^2+y^2\right)^{-1/2}\left(2x\frac{dx}{dt}+2y\frac{dy}{dt}\right)$$
$$=\frac{1}{\sqrt{x^2+y^2}}\left(x\frac{dx}{dt}+y\frac{dy}{dt}\right)$$
$$=\frac{x}{\sqrt{x^2+y^2}}\frac{dx}{dt}+\frac{y}{\sqrt{x^2+y^2}}\frac{dy}{dt}$$

c) $$s=\sqrt{x^2+y^2}\quad\Rightarrow\quad s^2=x^2+y^2$$
$$0=2x\frac{dx}{dt}+2y\frac{dy}{dt}$$
$$2x\frac{dx}{dt}=-2y\frac{dy}{dt}$$
$$\frac{dx}{dt}=-\frac{y}{x}\frac{dy}{dt}$$

Exercise

A 13-ft ladder is leaning against a house when its base starts to slide away. By the time the base is 12 *ft* from the house, the base is moving at the rate of 5 *ft/sec*.



a) How fast is the top of the ladder sliding down the wall then?

b) At what rate is the area of the triangle formed by the ladder, wall, and the ground changing then?

c) At what rate is the angle θ between the ladder and the ground changing then?

Solution

Given: $L = 13\ ft \quad x = 12 \quad \frac{dx}{dt} = 5\ ft/\sec$

$$y = \sqrt{13^2 - 12^2} = 5$$

a) $x^2 + y^2 = 13^2$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$y\frac{dy}{dt} = -x\frac{dx}{dt}$$

$$\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}$$

$$= -\frac{12}{5}(5)$$

$$\underline{= -12\ ft/\sec}$$ The ladder is sliding down the wall

b) Area of the triangle formed by the ladder and the walls is: $A = \frac{1}{2}xy$

$$\frac{dA}{dt} = \frac{1}{2}\left(y\frac{dx}{dt} + x\frac{dy}{dt}\right)$$

$$= \frac{1}{2}\left((5)(5) + (12)(-12)\right)$$

$$\underline{= -19.5\ ft^2/\sec}$$

c) $\cos\theta = \frac{x}{13} \quad \Rightarrow \quad -\sin\theta\frac{d\theta}{dt} = \frac{1}{13}\frac{dx}{dt}$

$$\frac{d\theta}{dt} = -\frac{1}{13\sin\theta}\frac{dx}{dt}$$

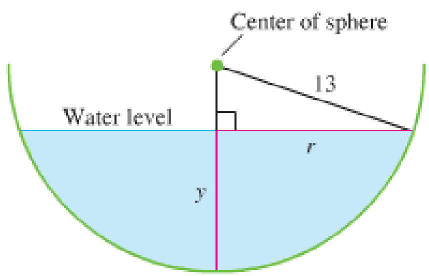
$$= -\frac{1}{13\sin\theta}(5) \qquad \sin\theta = \frac{5}{13}$$

$$= -\frac{1}{13\left(\frac{5}{13}\right)}(5)$$

$$\underline{= -1\ rad/\sec}$$

Exercise

Water is flowing at the rate of 6 m^3 / min from a reservoir shaped like a hemispherical bowl of radius 13 ***m***. Answer the following questions, given that the volume of water in a hemispherical bowl of radius R is $V = \frac{\pi}{3} y^2 (3R - y)$ when the water is y meters deep.



a) At what rate the water level changing when the water is 8 ***m*** deep?

b) What is the radius r of the water's surface when the water is y ***m*** deep?

c) At what rate is the radius r changing when the water is 8 ***m*** deep?

Solution

Given: $\frac{dV}{dt} = 6\ m^3 / \min \quad R = 13\ m$

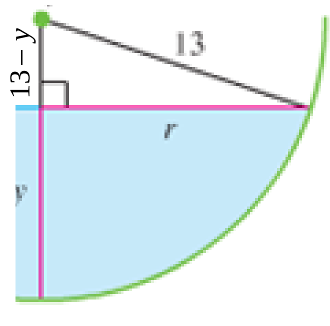
a) $V = \frac{\pi}{3} y^2 (3R - y) = \pi R y^2 - \frac{\pi}{3} y^3$

$$\frac{dV}{dt} = \left(2\pi R y - \pi y^2\right) \frac{dy}{dt}$$ ***Factor*** πy

$$\frac{dV}{dt} = \pi y (2R - y) \frac{dy}{dt}$$

$$\frac{dy}{dt} = \frac{1}{\pi(8)\left(2(13) - (8)\right)}(-6) = -\frac{1}{24\pi}\ m / \min$$

b) The hemispherical is on the circle: $r^2 + (13 - y)^2 = 13^2$



$$r^2 = 169 - \left(169 - 26y + y^2\right)$$

$$= 169 - 169 + 26y - y^2$$

$$= 26y - y^2$$

$$\boxed{r = \sqrt{26y - y^2}}$$

c) $r = \left(26y - y^2\right)^{1/2} \quad \Rightarrow \quad \frac{dr}{dt} = \frac{1}{2}\left(26y - y^2\right)^{-1/2} (26 - 2y) \frac{dy}{dt}$

$$= \frac{1}{2} \frac{26 - 2y}{\sqrt{26y - y^2}} \frac{dy}{dt}$$

$$\left.\frac{dr}{dt}\right|_{y=8} = \frac{1}{2} \frac{26 - 2(8)}{\sqrt{26(8) - (8)^2}} \left(-\frac{1}{24\pi}\right) = -\frac{5}{288\pi} = 0.005526 \quad or \quad \boxed{5.526 \times 10^{-3}}$$

Exercise

A spherical balloon is inflated with helium at the rate of $100\pi\ ft^3/\min$. How fast is the balloon's radius increasing at the instant the radius is 5 ft? How fast the surface area increasing?

Solution

Given: $\frac{dV}{dt} = 100\pi\ ft^3/\min \quad r = 5\ ft$

If $V = \frac{4}{3}\pi r^3 \quad \Rightarrow \quad \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

$$\frac{dr}{dt} = \frac{1}{4\pi r^2}\frac{dV}{dt}$$

$$= \frac{1}{4\pi(5)^2}(100\pi)$$

$$= 1\ ft/\min$$

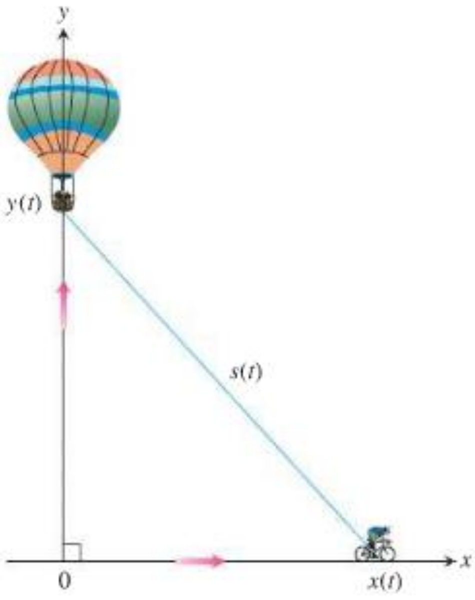
$$S = 4\pi r^2 \quad \Rightarrow \quad \frac{dS}{dt} = 8\pi r\frac{dr}{dt} = 8\pi(5)(1) = 40\pi\ ft^2/\min$$

The rate of the surface area is increasing.

Exercise

A balloon rising vertically above a level, straight road at a constant rate of 1 ft/sec. Just when the balloon is 65 ft above the ground, a bicycle moving at a constant rate of 17 ft/sec passes under it. How fast is the distance $s(t)$ between the bicycle and the balloon increasing 3 sec later?

Solution



Given: $\frac{dy}{dt} = 1\ ft/\sec \quad y = 65\ ft \quad \frac{dx}{dt} = 17\ ft/\sec$

Bicycle increasing 3 sec: $x = vt = 17(3) = 51\ ft$

$$s^2 = x^2 + y^2 \quad \Rightarrow \quad s = \sqrt{51^2 + 65^2} \approx 83\ ft$$

$$2s\frac{ds}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

$$\frac{ds}{dt} = \frac{1}{s}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right)$$

$$= \frac{1}{83}\big(51(17) + 65(1)\big)$$

$$\approx 11\ ft/\sec$$

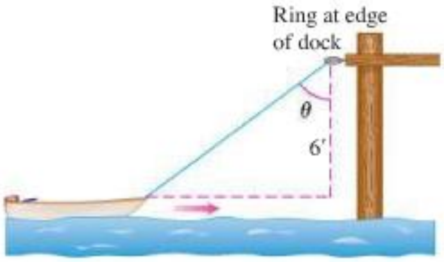
Exercise

A dinghy is pulled toward a dock by a rope from the bow through a ring on the dock 6 *ft* above the bow. The rope is hauled in at rate of 2 *ft/sec*.

a) How fast is the boat approaching the dock when 10 ft of rope are out?

b) At what rate is the angle θ changing at this instant?

Solution



Given: $h = 6\ ft \quad \frac{ds}{dt} = -2\ ft/\sec$

a) $s = 10\,ft$

$$s^2 = x^2 + 6^2 \quad \Rightarrow \quad x = \sqrt{s^2 - 36}$$

$$2s\frac{ds}{dt} = 2x\frac{dx}{dt}$$

$$\frac{dx}{dt} = \frac{s}{\sqrt{s^2 - 36}}\frac{ds}{dt}$$

$$\left.\frac{dx}{dt}\right|_{s=10} = \frac{10}{\sqrt{10^2 - 36}}(-2) = \underline{-2.5\ ft/\sec}$$

b) $\cos\theta = \frac{6}{s} \quad \Rightarrow \quad -\sin\theta\frac{d\theta}{dt} = -\frac{6}{s^2}\frac{ds}{dt}$

$$\frac{d\theta}{dt} = \frac{6}{\sin\theta s^2}\frac{ds}{dt} \qquad \sin\theta = \frac{x}{s} = \frac{\sqrt{10^2 - 36}}{10} = \frac{8}{10}$$

$$\frac{d\theta}{dt} = \frac{6}{(.8)10^2}(-2) = \underline{-0.15\ rad/\sec}$$

Exercise

The coordinates of a particle in the metric *xy*–plane are differentiable functions of time *t* with $\frac{dx}{dt} = -1\ m/\sec$ and $\frac{dy}{dt} = -5\ m/\sec$. How fast is the particle's distance from the origin changing as it passes through the point (5, 12)?

Solution

Given: $\frac{dx}{dt} = -1\ m/\sec \quad \frac{dy}{dt} = -5\ m/\sec$

$$s^2 = x^2 + y^2 \qquad \Rightarrow \quad s = \sqrt{x^2 + y^2} = \sqrt{5^2 + 12^2} = 13$$

$$2s\frac{ds}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

$$\frac{ds}{dt} = \frac{1}{s}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right)$$

$$\left.\frac{ds}{dt}\right|_{(5,12)} = \frac{1}{13}\big(5(-1) + 12(-5)\big) = \underline{-5\ m/\sec}$$

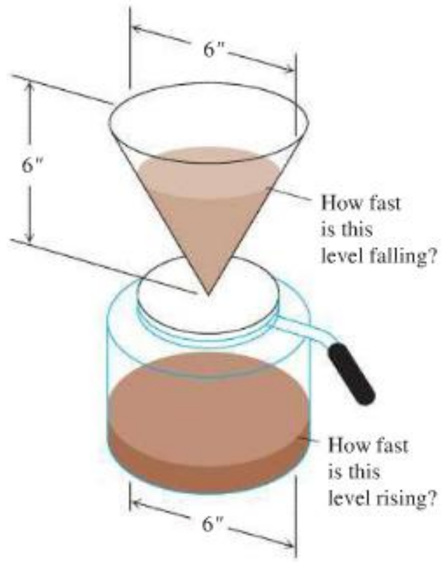
Exercise

Coffee is draining from a conical filter into a cylindrical coffeepot at the rate of 10 in^3 / min .

a) How fast is the level in the pot rising when the coffee in the cone is 5 in. deep?

b) How fast is the level in the cone falling then?

Solution



$$r_{pot} = 3 \quad \frac{dV}{dt} = 10 \ in^3 / \min$$

a) Let h be the height of the coffee in the pot.

Volume of the coffee: $V = \pi r^2 h = 9\pi h$

$$\frac{dV}{dt} = 9\pi \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{1}{9\pi}\frac{dV}{dt} = \frac{1}{9\pi}(10) = \frac{10}{9\pi} \ in / \min$$

b) Radius of the filter: $r = \frac{h}{2}$

Volume of the filter: $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{\pi h^3}{12}$

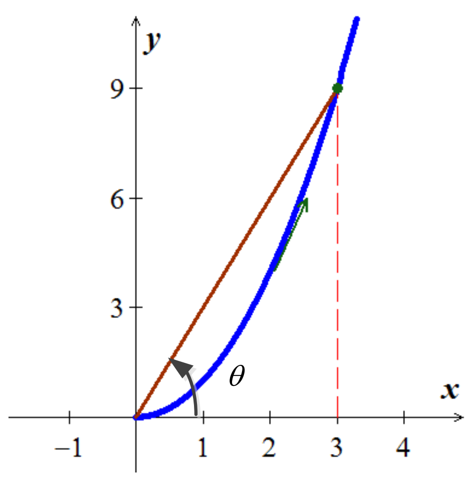
$$\frac{dV}{dt} = \frac{\pi}{4}h^2 \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{4}{\pi h^2}\frac{dV}{dt} = \frac{4}{\pi(5)^2}(-10) = -\frac{8}{5\pi} \ in / \min$$

Exercise

A particle moves along the parabola $y = x^2$ in the first quadrant in such a way that its *x*-coordinate (measure in meters) increases at a steady 10 m/sec. How fast is the angle of inclination θ of the line joining the particle to the origin changing when *x* = 3 *m*?

Solution



Given: $y = x^2 \quad v = \frac{dx}{dt} = 10 \ m / \sec \quad x = 3 \ m$

$$\tan\theta = \frac{y}{x} = \frac{x^2}{x} = x$$

$$\frac{d}{dt}\tan\theta = \frac{d}{dt}x$$

$$\sec^2\theta \frac{d\theta}{dt} = \frac{dx}{dt}$$

$$\frac{d\theta}{dt} = \frac{1}{\sec^2\theta}\frac{dx}{dt} = \cos^2\theta \frac{dx}{dt}$$

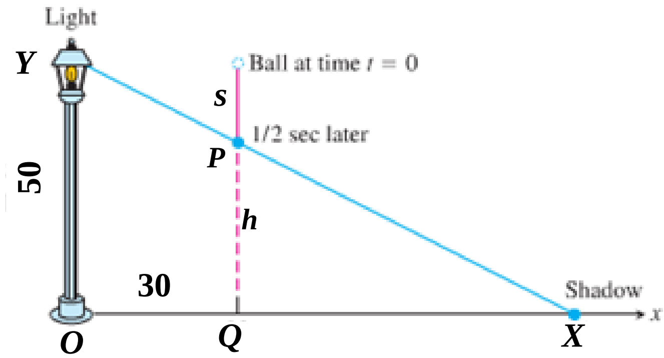
$$= \left(\frac{3}{\sqrt{9^2 + 3^2}}\right)^2 (10)$$

$$= 1 \ rad / \sec$$

Exercise

A light shines from the top of a pole 50 *ft* high. A ball is dropped from the same height from a point 30 ft away from the light. How fast is the shadow of the ball moving along the ground $\frac{1}{2}$ *sec* later? (Assume the ball falls a distance $s=16t^2$ ft in t sec.)

Solution



$$s=16t^2$$
$$s+h=50$$

Triangles XOY and XQP are similar:

$$\therefore \frac{XQ}{h}=\frac{OX}{50}=\frac{30+XQ}{50}$$

$$50|XQ|=30h+h|XQ|$$

$$(50-h)|XQ|=30h$$

$$|XQ|=\frac{30h}{50-h}$$

$$=\frac{30(50-s)}{50-(50-s)}$$

$$=\frac{30\left(50-16t^2\right)}{50-50+16t^2}$$

$$=\frac{1500-480t^2}{16t^2}$$

$$=\frac{1500}{16t^2}-\frac{480t^2}{16t^2}$$

$$=\frac{1500}{16t^2}-30$$

$$\frac{d}{dt}|XQ|=1500\frac{-32t}{\left(16t^2\right)^2}$$

$$=1500\frac{-32t}{256t^4}$$

$$=-\frac{375}{2t^3}$$

$$\left.\frac{d}{dt}|XQ|\right|_{t=\frac{1}{2}}=-\frac{375}{2\left(\frac{1}{2}\right)^3}=\underline{-1500\ \ ft/\sec}$$

Exercise

A spherical iron ball 8 in. in diameter is coated with a layer of ice of uniform thickness. If the ice melts at the rate of 10 $in^3/\min$, how fast is the thickness of the ice decreasing when it is 2 in. thick? How fast is the outer surface area of ice decreasing?

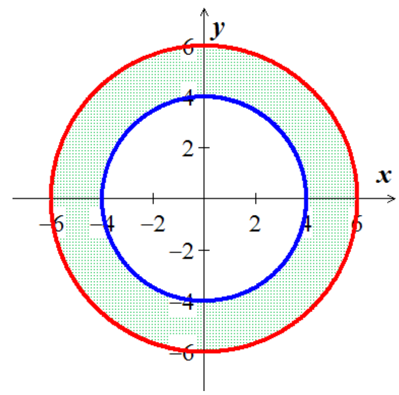
Solution

Given: $D = 8\ in \to r_1 = 4\ in \quad \frac{dV}{dt} = -10\ \text{in}^3/\min \quad think = 2\ \text{in}$

$$V = \frac{4}{3}\pi r^3 \quad \Rightarrow \quad \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{1}{4\pi r^2}\frac{dV}{dt}$$

$$\left.\frac{dr}{dt}\right|_{r=6} = \frac{1}{4\pi(6)^2}(-10) = -\frac{5}{72\pi}\ in/\min$$



$$S = 4\pi r^2$$

$$\frac{dS}{dt} = 8\pi r\frac{dr}{dt}$$

$$\left.\frac{dS}{dt}\right|_{r=6} = 8\pi(6)\left(-\frac{5}{72\pi}\right) = \underline{-\frac{10}{3}\ in^2/\min}$$

The outer surface are of the ice is decreasing at $-\frac{10}{3}\ in^2/\min$

Exercise

On a morning of a day when the sun will pass directly overhead, the shadow of an 80–*ft* building on level ground is 60 *ft* long. At the moment in question, the angle θ the sun makes with the ground is increasing at the rate of 0.27 °/ min. At what rate is the shadow decreasing?

Solution

Given: $x = 60\ ft \quad h = 80\ ft$

$$\frac{d\theta}{dt} = 0.27°\min = 0.27°\frac{\pi rad}{180°}\frac{1}{\min} = \frac{3\pi}{2000} rad/\min$$

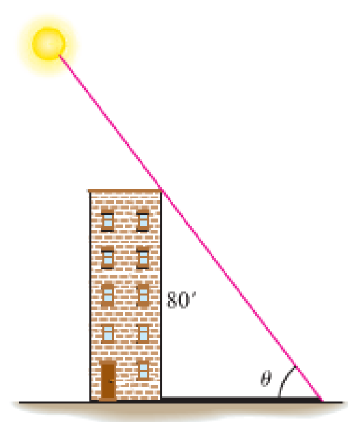
$$\tan\theta = \frac{80}{x} \quad \Rightarrow \quad \frac{d}{dt}\tan\theta = \frac{d}{dt}\frac{80}{x}$$

$$\sec^2\theta\frac{d\theta}{dt} = -\frac{80}{x^2}\frac{dx}{dt}$$

$$\frac{dx}{dt} = \left|-\frac{x^2\sec^2\theta}{80}\frac{d\theta}{dt}\right| \qquad \cos\theta = \frac{60}{\sqrt{60^2+80^2}} = \frac{60}{100} = \frac{3}{5}$$

$$= \frac{60^2\left(\frac{5}{3}\right)^2}{80}\left(\frac{3\pi}{2000}\right)$$

$$\underline{= 0.589\ ft/\min}$$



Exercise

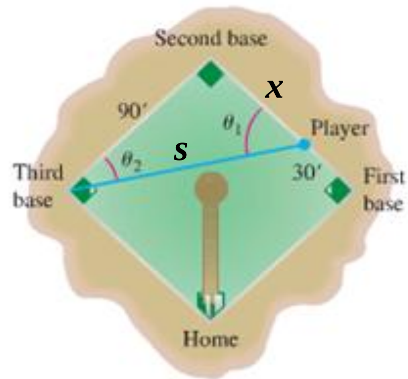
A baseball diamond is a square 90 *ft* on a side. A player runs from first base to second at a rate of 16 ft/sec.

a) At what rate is the player's distance from third base changing when the player is 30 ft from first base?

b) At what rates are angles θ_1 *and* θ_2 changing at that time?

c) The player slides into second base at the rate of 15 ft/sec. At what rates are angles θ_1 *and* θ_2 changing as the player touches base?

Solution



Given: $d_1 = 90\ ft \quad d_2 = 30\ ft \quad \frac{dx}{dt} = -16\ ft/\sec$

x: Distance between player and 2^{nd} base

s: Distance between player and 3^{rd} base

a) $x = 90 - 30 = 60\ ft$

$$s^2 = x^2 + 90^2 \quad \to \quad s = \sqrt{60^2 + 90^2} = \sqrt{11700} = 30\sqrt{13}$$

$$2s\frac{ds}{dt} = 2x\frac{dx}{dt}$$

$$\frac{ds}{dt} = \frac{x}{s}\frac{dx}{dt}$$

$$= \frac{60}{30\sqrt{13}}(-16)$$

$$\approx -8.875\ ft/\sec$$

b) $\sin\theta_1 = \frac{90}{s} \quad \to \quad \cos\theta_1 \frac{d\theta_1}{dt} = -\frac{90}{s^2}\frac{ds}{dt}$

$$\frac{d\theta_1}{dt} = -\frac{90}{s^2\cos\theta_1}\frac{ds}{dt} \qquad \cos\theta_1 = \frac{x}{s}$$

$$\frac{d\theta_1}{dt} = -\frac{90}{s^2\frac{x}{s}}\frac{ds}{dt}$$

$$= -\frac{90}{s \cdot x}\frac{ds}{dt}$$

$$= -\frac{90}{30\sqrt{13}(60)}(-8.875)$$

$$\approx 0.123\ rad/\sec$$

$$\cos\theta_2 = \frac{90}{s} \quad \to \quad -\sin\theta_2\frac{d\theta_2}{dt} = -\frac{90}{s^2}\frac{ds}{dt}$$

$$\frac{d\theta_2}{dt} = \frac{90}{s^2\sin\theta_2}\frac{ds}{dt} = \frac{90}{s \cdot x}\frac{ds}{dt} \qquad \sin\theta_2 = \frac{x}{s}$$

$$= \frac{90}{30\sqrt{13}(60)}(-8.875)$$

$$\approx -0.123 \ rad / \sec$$

c) $$\frac{d\theta_1}{dt} = -\frac{90}{s^2 \cos\theta_1}\frac{ds}{dt}$$ $$\frac{ds}{dt} = \frac{x}{s}\frac{dx}{dt}$$

$$= -\frac{90}{s^2 \frac{x}{s}}\frac{x}{s}\frac{dx}{dt}$$

$$= -\frac{90}{s^2}\frac{dx}{dt}$$

$$= -\frac{90}{x^2 + 8100}\frac{dx}{dt}$$

Player slides into second base $\Rightarrow x = 0$

$$\left.\frac{d\theta_1}{dt}\right|_{x=0} = -\frac{90}{0^2 + 8100}(-15) = \frac{1}{6} \ rad / \sec$$

$$\frac{d\theta_2}{dt} = \frac{90}{s^2 \sin\theta_2}\frac{ds}{dt} = \frac{90}{s^2 \frac{x}{s}}\frac{x}{s}\frac{dx}{dt} = \frac{90}{s^2}\frac{dx}{dt}$$

$$= \frac{90}{x^2 + 8100}\frac{dx}{dt}$$

Player slides into second base $\Rightarrow x = 0$

$$\left.\frac{d\theta_2}{dt}\right|_{x=0} = \frac{90}{0^2 + 8100}(-15) = -\frac{1}{6} \ rad / \sec$$